

Solutions of Green's function for Lamb's problem of a two-phase saturated medium

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Abstract The solutions of Green's function are significant for simplification of problem on a two-phase saturated medium. Using transformation of axisymmetric cylindrical coordinate and Sommerfeld's integral, superposition of the influence field on a free surface, authors obtained the solutions of a two-phase saturated medium subjected to a concentrated force on the semi-space. © 2011 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1105203]

Keywords Green's function, two-phase saturated medium, Lamb's problem

Lamb's solutions have been widely used to solve response and vibration problems of soil dynamics and earthquake engineering.^{1,2} Utilizing that the correlative determinant of the eigen equations equals to zero, Phillipapoulos³ presented a group of closed-form analytical solutions for a two-phase saturated medium subjected to a concentrated force in the semi-space (Lamb's problem of a two-phase medium). Phillipapoulos's solutions were interested by the researchers, but process of his deduction is groping due to that coupling of fast and slow dilational waves had not been solved. Today a lot of dynamic solutions of a two-phase saturated medium subjected to a load in the semi-space have to be solved (other Lamb's problem).⁴ And it is needful to concisely and regularly deduct on the dynamic problems in the semi-space of a two-phase saturated medium.

Decoupling potential field of fast and slow dilational wave on dynamic equations of a two-phase saturated medium,⁵⁻⁸ authors obtained Green's functions of an infinite space subjected to a concentrated force.⁸ Through transformation of cylindrical coordinate and Sommerfeld's integral, as well as superposition of the influence function on a free surface, the dynamic solutions of a two-phase saturated medium subjected to a concentrated force in the semi-space had attained in this paper. These results are consistent with Phillipapoulos's solution.³ While a two-phase medium degenerates into a single phase medium, these results are consistent with Lamb's solutions.⁹ The process of deduction is clear and straightforward. Hence, it can be beneficial to solving pertinent dynamic problems in the semi-space of a two-phase saturated medium.

Biot's dynamic equation of the two-phase saturated medium can be written as^{10,11}

$$\begin{aligned} \mu \nabla^2 \mathbf{u} + \nabla[(\lambda_c + \mu) \nabla \cdot \mathbf{u} + \alpha M \nabla \cdot \mathbf{w}] = \\ \rho_f \ddot{\mathbf{u}} + \rho_f \ddot{\mathbf{w}}, \\ \nabla(\alpha \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}) = \rho_f \ddot{\mathbf{u}} + \gamma(\omega) \ddot{\mathbf{w}}, \\ \lambda_c = \lambda + \alpha M, \end{aligned} \quad (1)$$

where λ and μ are Lamé coefficients; α and M are the parameters derived by Biot in the research of a two-phase saturated medium; \mathbf{u}, \mathbf{w} are the vectors of the solid phase and the fluid phase of displacement field, respectively; ρ, ρ_f are the substance density of the two-phase medium and the fluid phase medium, respectively. If we suppose that k_s, k_f and k_b are the solid, fluid and two-phase bulk moduli, respectively, β_0 is the porosity, then we have¹¹

$$\begin{aligned} \lambda_c &= (k_s - k_b)^2 / (D - k_b) + k_b - 2\mu/3, \\ \alpha &= (k_s - k_b) / k_s, \\ M &= k_s^2 / (D - k_b), \\ D &= k_s [1 + (k_s / k_f - 1) \beta_0], \end{aligned} \quad (2)$$

where $\lambda_c, \mu, \alpha, M$ can be regarded as absolute elastic constants. The $\gamma(\omega)$ is the dissipation coefficient. When frequency $\omega < \omega_c$ (ω_c is cut-off frequency, $\omega_c = 0.06\pi k_d \rho_f / \eta \beta_0 \approx 6.28 \times 10^6 \cdot \text{s}^{-1}$, η is the viscosity coefficient; k_d is the penetration coefficient), $\gamma(\omega)$ becomes a constant γ . The solution of Green's function for dynamic equation is⁸

$$\begin{aligned} G(\mathbf{x}/\boldsymbol{\varsigma}, \omega) = \frac{1}{4\pi\omega^2} \{ \eta_3 [\nabla \times \nabla \times \mathbf{I}(e^{-iK_\beta R}/R)] - \\ \eta_1 [\nabla \nabla \cdot \mathbf{I}(e^{-iK_{\alpha_1} R}/R)] + \\ \eta_2 [\nabla \nabla \cdot \mathbf{I}(e^{-iK_{\alpha_2} R}/R)] \}, \end{aligned} \quad (3)$$

where $\eta_1 = \lambda_1 / (\rho + \rho_f \xi_1)$, $\eta_2 = \lambda_2 / (\rho + \rho_f \xi_1)$, $\eta_3 = 1 / (\rho - \rho_f^2 / \gamma)$, which represent the participant parameters of mass for fast and slow dilational waves, and the distortional wave, respectively. Here $\lambda_n = (1 + \xi_n) / (\xi_1 - \xi_2)$, ($n = 1, 2$), ξ_1 and ξ_2 are the dynamic parameters, $\xi_n = (\lambda_c + 2\mu - \rho \alpha_n^2) / (\rho_f \alpha_n^2 - \alpha M)$, α_1, α_2 and β are the velocities of fast, slow dilational and distortional waves of a two-phase saturated medium, respectively. The $R = |\mathbf{R}| = |\mathbf{x} - \boldsymbol{\varsigma}|$ is the distance from the source point $\boldsymbol{\varsigma}$ to the field point \mathbf{x} . The $K_{\alpha_1} = \omega / \alpha_1$, $K_{\alpha_2} = \omega / \alpha_2$ and $K_\beta = \omega / \beta$ are the wave numbers of fast, slow dilational and distortional waves, respectively. The \mathbf{I} is a second order unit tensor, and $\mathbf{G}(\mathbf{x}/\boldsymbol{\varsigma}, \omega)$ is a second order tensor of Green's function in the frequency domain.

Take a cylindrical coordinate (r, θ, z) , where $x =$

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$x_1 - \varsigma_1 = r \cos \theta$, $y = x_2 - \varsigma_2 = r \sin \theta$, $z = x_3 - \varsigma_3$; $R^2 = r^2 + z^2$, r is the radial vector of a cylindrical coordinate.

Equation (3) on an axisymmetric cylindrical coordinate can be written as

$$4\pi\omega^2 G_{ki}(r, \omega) = \eta_3 \begin{pmatrix} -r^{-1}\partial^2/\partial z^2 & 0 & r^{-1}\partial^2/(\partial r\partial z) \\ 0 & r^{-2} - \nabla^2 & 0 \\ r^{-1}\partial^2/(\partial z\partial r) & 0 & -r^{-1}\partial^2/\partial r^2 \end{pmatrix} e^{-iK_\beta R}/R -$$

$$\eta_1 \begin{pmatrix} \nabla^2 - r^{-1}\partial^2/\partial z^2 & 0 & r^{-1}\partial^2/(\partial r\partial z) \\ 0 & r^{-2} & 0 \\ r^{-1}\partial^2/(\partial z\partial r) & 0 & \nabla^2 - r^{-1}\partial^2/\partial r^2 \end{pmatrix} e^{-iK_{\alpha_1} R}/R +$$

$$\eta_2 \begin{pmatrix} \nabla^2 - r^{-1}\partial^2/\partial z^2 & 0 & r^{-1}\partial^2/(\partial r\partial z) \\ 0 & r^{-2} & 0 \\ r^{-1}\partial^2/(\partial z\partial r) & 0 & \nabla^2 - r^{-1}\partial^2/\partial r^2 \end{pmatrix} e^{-iK_{\alpha_2} R}/R. \quad (4)$$

By Sommerfeld's formula: $e^{-iK_c R}/R = \int_0^R J_0(kr) e^{-cz} k dk / c$, ($c = \sqrt{k^2 - K_c^2}$, where K_c can be arbitrary one of K_{α_1} , K_{α_2} and K_β), c is equal to one of a_1 , a_2 , b corresponding to α_1 , α_2 , β respectively. From property of Bessel's function, we can attain

$$u_r = G_{zr} = \frac{1}{4\pi\omega^2 r} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) k J_1(kr) k dk, \quad (5)$$

$$u_z = G_{zz} = \frac{1}{4\pi\omega^2 r} \int_0^\infty \left[\eta_3 \frac{e^{-bz}}{b} - \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) \frac{e^{-a_1 z}}{a_1} + \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) \frac{e^{-a_2 z}}{a_2} \right] k^2 J_0(kr) k dk -$$

$$\frac{1}{4\pi\omega^2 r^2} \int_0^\infty \left(\eta_3 \frac{e^{-bz}}{b} - \eta_1 \frac{e^{-a_1 z}}{a_1} + \eta_2 \frac{e^{-a_2 z}}{a_2} \right) k J_1(kr) k dk. \quad (6)$$

The constitutive formula on an axisymmetric cylindrical coordinate is

$$\sigma_r = \lambda e + 2\mu \frac{\partial u_r}{\partial r}, \quad \sigma_z = \lambda e + 2\mu \frac{\partial u_z}{\partial z}, \quad \tau_{rz} = \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right), \quad (7)$$

where e is the bulk strain. Substituting Eqs. (5) and (6) into Eq. (7), we obtain

$$\sigma_z = \frac{\lambda}{4\pi\omega^2 r} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) k^2 J_0(kr) k dk +$$

$$\frac{(\lambda + 2\mu)}{4\pi\omega^2 r} \int_0^\infty \left[-\eta_3 e^{-bz} + \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) e^{-a_1 z} - \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) e^{-a_2 z} \right] k^2 J_0(kr) k dk +$$

$$\frac{2\mu}{4\pi\omega^2 r^2} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) J_1(kr) k dk, \quad (8)$$

$$\tau_{rz} = \frac{\mu}{4\pi\omega^2 r} \int_0^\infty (-\eta_3 e^{-bz} + a_1 \eta_1 e^{-a_1 z} - a_2 \eta_2 e^{-a_2 z}) J_1(kr) k^2 dk -$$

$$\frac{\mu}{4\pi\omega^2 r} \int_0^\infty \left[\eta_3 \frac{e^{-bz}}{b} - \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) \frac{e^{-a_1 z}}{a_1} + \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) \frac{e^{-a_2 z}}{a_2} \right] k^3 J_1(kr) k dk +$$

$$\frac{3\mu}{4\pi\omega^2 r^3} \int_0^\infty \left(\eta_3 \frac{e^{-bz}}{b} - \eta_1 \frac{e^{-a_1 z}}{a_1} + \eta_2 \frac{e^{-a_2 z}}{a_2} \right) J_1(kr) k^2 dk. \quad (9)$$

On the semi-infinite space ($z > 0$, the positive z axis is downward), the forms of the influence function of a free surface can be assumed as

$$u_{r,z}^0 = f_1(e^{-a_1 z}, e^{-a_2 z}, e^{-bz}) J_0(kr) k dk + f_2(e^{-a_1 z}, e^{-a_2 z}, e^{-bz}) J_1(kr) k dk, \quad (10)$$

According to Eqs. (8) and (9), the influence function of the free surface can be considered in detail as

$$u_r^0 = \frac{1}{4\pi\mu} \int_0^\infty (Ae^{-a_1 z} + Be^{-a_2 z} + Ce^{-bz}) J_1(kr) k^2 dk +$$

$$\frac{D}{4\pi\omega^2 r} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) J_1(kr) k^2 dk. \quad (11)$$

$$\begin{aligned}
u_z^0 = & \frac{1}{4\pi\mu} \int_0^\infty \left(a_1 A e^{-a_1 z} + a_2 B e^{-a_2 z} + C k^2 \frac{e^{-bz}}{b} \right) J_0(kr) k dk + \\
& E \frac{1}{4\pi\omega^2 r} \int_0^\infty \left[\eta_3 \frac{e^{-bz}}{b} - \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) \frac{e^{-a_1 z}}{a_1} + \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) \frac{e^{-a_2 z}}{a_2} \right] J_0(kr) k^3 dk - \\
& \frac{F}{4\pi\omega^2 r^2} \int_0^\infty \left(\eta_3 \frac{e^{-bz}}{b} - \eta_1 \frac{e^{-a_1 z}}{a_1} + \eta_2 \frac{e^{-a_2 z}}{a_2} \right) J_1(kr) k^2 dk,
\end{aligned} \quad (12)$$

where A, B, C, D, E are the undetermined parameters. Substituting Eqs. (11) and (12) into Eq. (7), we obtain

$$\begin{aligned}
\sigma_z^0 = & \frac{1}{4\pi\mu} \int_0^\infty \lambda \left[(A e^{-a_1 z} + B e^{-a_2 z} + C e^{-bz}) k - (\lambda + 2\mu) (a_1^2 A e^{-a_1 z} + a_2^2 B e^{-a_2 z} + k^2 C e^{-bz}) \right] J_0(kr) k^2 dk + \\
& D \frac{\lambda}{4\pi\omega^2 r} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) J_0(kr) k^3 dk - \\
& D \frac{\lambda}{4\pi\omega^2 r^2} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) J_1(kr) k^2 dk + \\
& E \frac{(\lambda + 2\mu)}{4\pi\omega^2 r} \int_0^\infty \left[-\eta_3 e^{-bz} + \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) e^{-a_1 z} - \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) e^{-a_2 z} \right] J_0(kr) k^3 dk + \\
& F \frac{(\lambda + 2\mu)}{4\pi\omega^2 r^2} \int_0^\infty (\eta_3 e^{-bz} - \eta_1 e^{-a_1 z} + \eta_2 e^{-a_2 z}) J_1(kr) k^2 dk,
\end{aligned} \quad (13)$$

$$\begin{aligned}
\tau_{rz}^0 = & -\frac{\mu}{4\pi\mu} \int_0^\infty \left(a_1 A e^{-a_1 z} + a_2 B e^{-a_2 z} + b C e^{-bz} + a_1 A e^{-a_1 z} + a_2 B e^{-a_2 z} + k^2 C \frac{e^{-bz}}{b} \right) J_1(kr) k^2 dk - \\
& D \frac{\mu}{4\pi\omega^2 r} \int_0^\infty (\eta_3 b e^{-bz} - \eta_1 a_1 e^{-a_1 z} + \eta_2 a_2 e^{-a_2 z}) J_1(kr) k^2 dk - \\
& \mu E \frac{1}{4\pi\omega^2 r} \int_0^\infty \left[\eta_3 \frac{e^{-bz}}{b} - \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) \frac{e^{-a_1 z}}{a_1} + \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) \frac{e^{-a_2 z}}{a_2} \right] J_1(kr) k^3 dk + \\
& F \frac{3\mu}{4\pi\omega^2 r^3} \int_0^\infty \left(\eta_3 \frac{e^{-bz}}{b} - \eta_1 \frac{e^{-a_1 z}}{a_1} + \eta_2 \frac{e^{-a_2 z}}{a_2} \right) J_1(kr) k^2 dk - \\
& \mu F \frac{1}{4\pi\omega^2 r^2} \int_0^\infty \left(\eta_3 \frac{1}{b} e^{-bz} - \eta_1 \frac{1}{a_1} e^{-a_1 z} + \eta_2 \frac{1}{r^2 a_2} e^{-a_2 z} \right) k^2 J_0(kr) k dk - \\
& E \frac{\mu}{4\pi\omega^2 r^2} \int_0^\infty \left(\eta_3 \frac{e^{-bz}}{b} - \eta_1 \frac{e^{-a_1 z}}{a_1} + \eta_2 \frac{e^{-a_2 z}}{a_2} \right) J_0(kr) k^3 dk.
\end{aligned} \quad (14)$$

When a concentrated force at a surface is $\int_0^\infty P_0 J_0(kr) k dk / 2\pi$,³ according to the boundary condition of a free surface ($z = 0$), we have

$$(\sigma_z^0 + \sigma_z)|_{z=0} - \alpha P_f = - \int_0^\infty \frac{P_0}{2\pi} J_0(kr) k dk, \quad (\tau_{rz}^0 + \tau_{rz}^0)|_{z=0} = 0, \quad P_f(r, 0)|_{z=0} = 0. \quad (15)$$

Substituting Eqs. (8), (9), (13) and (14) into Eq. (15), we obtain following set of undetermined parameters

$$\begin{aligned}
& \frac{\lambda}{4\pi\mu} \int_0^\infty \left[Ak + Bk + Ck - \frac{\lambda + 2\mu}{4} \pi \mu (a_1^2 A + a_2^2 B + k^2 C) + \frac{1}{2\pi} \right] J_0(kr) k^2 dk - \alpha P_f = 0, \\
& D(\eta_3 b - \eta_1 a_1 + \eta_2 a_2) = -(\eta_3 b - \eta_1 a_1 + \eta_2 a_2), \\
& E \left[-\eta_3 + \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) - \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) \right] = \left[\eta_3 - \eta_1 \left(1 - r + \frac{ra_1^2}{k^2} \right) + \eta_2 \left(1 - r + \frac{ra_2^2}{k^2} \right) \right], \\
& [(\lambda + 2\mu)F - D](\eta_3 - \eta_1 + \eta_2) = -(\eta_3 - \eta_1 + \eta_2).
\end{aligned} \quad (16)$$

From Eq. (16) we can show $D = E = F = -1$. Therefore, the displacement in radial direction U_r and the displacement in direction of Z axis U_z in the semi-space respectively are

$$U_r = u_r + u_r^0 = \frac{1}{4\pi\mu} \int_0^\infty (A k e^{-a_1 z} + B k e^{-a_2 z} + C k e^{-bz}) J_1(kr) k dk, \quad (17)$$

$$U_z = u_z + u_z^0 = \frac{1}{4\pi\mu} \int_0^\infty \left(a_1 e^{-a_1 z} + a_2 B e^{-a_2 z} + k^2 \frac{1}{b} C e^{-b z} \right) J_0(kr) k dk. \quad (18)$$

For the displacement of the fluid phase W_r , W_z in the semi-infinite space, taking the disability of transmission of distortional wave in the fluid phase into consideration and combining Eqs. (17) and (18), we get⁸

$$W_r = \frac{1}{4\pi\mu} \int_0^\infty (\xi_1 A k e^{-a_1 z} + \xi_2 B k e^{-a_2 z}) J_1(kr) k dk, \quad (19)$$

$$W_z = \frac{1}{4\pi\mu} \int_0^\infty (a_1 \xi_1 A e^{-a_1 z} + \xi_2 a_2 B e^{-a_2 z}) J_0(kr) k dk. \quad (20)$$

For the second equation of formulas of Eq. (1), we find

$$P(r, 0) = -\alpha M \left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z} \right) - M \left(\frac{\partial W_r}{\partial r} + \frac{W_r}{r} + \frac{\partial W_z}{\partial z} \right) \quad (21)$$

Equation (21) is reasonable to use in the semi-space also, so we have

$$-\alpha P_f(r, 0) = \frac{1}{4\pi\mu} \int_0^\infty [(\alpha^2 M + \alpha M \xi_1)(k^2 - a_1^2) \cdot A + (\alpha^2 M + \alpha M \xi_2)(k^2 - a_2^2) B] J_0(kr) k dk \quad (22)$$

By the boundary condition of a free surface $z = 0$, we obtain

$$\begin{aligned} & \frac{1}{4\pi\mu} [k(Ak + Bk + Ck) - (\lambda + 2\mu)\lambda \cdot \\ & (a_1^2 A + a_2^2 B + k^2 C) + 2\mu + \\ & (\alpha^2 M + \alpha M \xi_1)(k^2 - a_1^2) A + \\ & (\alpha^2 M + \alpha M \xi_2)(k^2 - a_2^2) B] = 0, \\ & (\text{for } \sigma_z^0 + \sigma_z|_{z=0} - \alpha P_f = \\ & - \int_0^\infty P_0 J_0(kr) k dk / 2\pi). \end{aligned} \quad (23)$$

$$\begin{aligned} & a_1 A + a_2 B + bC + a_1 A + a_2 B + k^2 \frac{1}{b} C = 0 \\ & (\text{for } \tau_{rz} + \tau_{rz}^0|_{z=0} = 0), \end{aligned} \quad (24)$$

$$\begin{aligned} & -P_f(r, z) = \alpha A k^2 e^{-a_1 z} + \alpha B k^2 e^{-a_2 z} - \\ & \alpha a_1^2 A e^{-a_1 z} - \alpha a_2^2 B e^{-a_2 z} + \xi_1 A k^2 e^{-a_1 z} + \\ & \xi_2 B k^2 e^{-a_2 z} - a_1^2 \xi_1 A e^{-a_1 z} - \\ & \xi_2 a_2^2 B e^{-a_2 z} = 0, \\ & (\text{for } P_f(r, 0) = 0). \end{aligned} \quad (25)$$

Using Eqs. (23), (24) and (25), we can arrive at

$$(\alpha + \xi_1)(k^2 - a_1^2) A + (\alpha + \xi_2)(k^2 - a_2^2) B = 0. \quad (26)$$

$$\begin{aligned} & \left[(a_1^2 - k^2) \frac{\lambda + \alpha^2 M + \alpha M \xi_1}{\mu} + 2a_1^2 \right] A + \\ & \left[(a_2^2 - k^2) \frac{\lambda + \alpha^2 M + \alpha M \xi_2}{\mu} + 2a_2^2 \right] B + \\ & 2k^2 C = 2, \\ & 2a_1 A + 2a_2 B + (b^2 + k^2) \frac{1}{b} C = 0, \\ & B = -\frac{(\alpha + \xi_1)(k^2 - a_1^2)}{(\alpha + \xi_2)(k^2 - a_2^2)} A. \end{aligned} \quad (27)$$

Let us recast Eq. (27) as

$$\begin{aligned} & k_1 A + k_2 B + 2k^2 C = 2, \\ & 2a_1 A + 2a_2 B + (b^2 + k^2) \frac{1}{b} C = 0, \end{aligned} \quad (28)$$

where

$$\begin{aligned} & B = \xi_4 A, \quad \xi_4 = \frac{(\alpha + \xi_1)(k^2 - a_1^2)}{(\alpha + \xi_2)(k^2 - a_2^2)}, \\ & k_{1,2} = \left[(a_{1,2}^2 - k^2) \frac{\lambda + \alpha^2 M + \alpha M \xi_{1,2}}{\mu} + 2a_{1,2}^2 \right], \\ & \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \xi_4 & 2k^2 \\ (2a_1 + 2a_2 \xi_4)b & b^2 + k^2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \end{aligned} \quad (29)$$

Marking the invertible matrix $(A)^{-1} = (A)^* / |A|$, we obtain

$$\begin{aligned} & \begin{pmatrix} k_1 + k_2 \xi_4 & 2k^2 \\ (2a_1 + 2a_2 \xi_4)b & b^2 + k^2 \end{pmatrix}^{-1} = \\ & \frac{1}{R} \begin{pmatrix} k_1 + k_2 \xi_4 & 2k^2 \\ (2a_1 + 2a_2 \xi_4)b & b^2 + k^2 \end{pmatrix}^*, \end{aligned} \quad (30)$$

where $(A)^*$ is the companion matrix of (A) and

$$R = (k_1 + k_2 \xi_4)(k^2 + b^2) - 4k^2(a_1 + a_2 \xi_4)b. \quad (31)$$

Equation (31) is a Rayleigh function. Substituting Eq. (30) into Eq. (32), we obtain

$$\begin{aligned} & \begin{pmatrix} A \\ C \end{pmatrix} = \frac{1}{R} \begin{pmatrix} k^2 + b^2 & -2k^2 \\ -2ab & k^2 + b^2 \end{pmatrix}^* \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \\ & \frac{1}{R} \begin{pmatrix} 2(k^2 + b^2) \\ -4(a_1 + a_2 \xi_4)b \end{pmatrix}. \end{aligned} \quad (32)$$

We substitute the obtained coefficients into the displacement expression Eqs. (17) and (18) to yield displacement solutions of a two-phase saturated medium

in the semi-space as

$$U_r|_{z=0} = \frac{1}{4\pi\mu} \int_0^\infty (Ak + Bk + Ck) J_1(kr) k dk = \int_0^\infty \frac{1}{2\pi\mu} \frac{k}{R} [(1 + \xi_4)(k^2 + b^2) - 2(a_1 + a_2\xi_4)b] J_1(kr) k dk, \quad (33)$$

$$U_z|_{z=0} = \frac{1}{4\pi\mu} \int_0^\infty \left(a_1 A + a_2 B + k^2 \frac{1}{b} C \right) J_0(kr) k dk = \frac{1}{2\pi\mu} \int_0^\infty \frac{1}{R} (a_1 + a_2\xi_4)(b^2 - k^2) J_0(kr) k dk. \quad (34)$$

Equations (33) and (34) are the displacement solutions in radial direction and the displacement solution in direction of Z axis in the semi-space, respectively, obtained by Philippacopoulos in 1988.⁸

When $\rho_f = 0, \xi_1 = 1, \xi_2 = 0$ a two-phase problem degenerates into a single-phase problem

$$U_z|_{z=0} = \frac{1}{2\pi\mu} \int_0^\infty \frac{a(k^2 - b^2)}{(k^2 + b^2)^2 - 4k^2 ab} J_0(kr) k dk, \quad (35)$$

$$U_r|_{z=0} = \frac{1}{2\pi\mu} \int_0^\infty \frac{(k^2 + b^2 - 2ab)}{(k^2 + b^2)^2 - 4k^2 ab} J_1(kr) k dk, \quad (36)$$

where $a = \sqrt{k^2 - K_a^2}$, K_a is the wave numbers of dilational wave. Equations (35) and (36) are Lamb's formulas.⁹

Based on the decoupling solutions of Green's function, through transformation of cylindrical coordinate and Sommerfeld's integral, as well as superposition of the influence function on a free surface, the method adopted to derive the result in this paper can be of further use in solving other Lamb's problems of a two-phase saturated medium. It can be regularly and straightforwardly beneficial to solving dynamic problems in semi-space.

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